

Continuity And Differentiability



A function f is said to have a left derivative at $x=c$ iff f is defined in some (undeleted) left neighborhood of c and $\lim_{h \rightarrow 0^-} \frac{f(c+h)-f(c)}{h}$ exists (finitely).

L.H.L

$$F'(x) = \lim_{h \rightarrow 0^-} \frac{F(c+h) - F(c)}{h}$$

R.H.D

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(c+h) - F(c)}{h}$$

Differentiability

Continuous

May be
May not

Derivative at any Point

A function f is said to have a derivative at any point x iff it is defined in some (undeleted) neighbourhood of the point x and $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ exists (finitely).

The value of the limits is called the derivative of f at any point x and is denoted by $f'(x)$ i.e

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}$$

Find the left hand and right hand derivatives of

$$f(x) = \begin{cases} x-3, & x < 2 \\ 2x-5, & x \geq 2 \end{cases} \text{ at } x=2 \text{ hence show that } f \text{ is not derivable at } x=2$$

$$\begin{array}{c} \text{L.H.D} \quad 2x-5 \\ \hline 2 \end{array}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

R.H.D

$$f(x) = 2x-5$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2(c+h)-5 - 2c+5}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{2c} + 2h - \cancel{5} - \cancel{2c} + \cancel{5}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h}{h} = 2$$

$$f'_-(2) \neq f'_+(2)$$

L.H.D

$$f(x) = x-3$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(c+h-3) - (c-3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{c} + h - \cancel{3} - \cancel{c} + \cancel{3}}{h} = 1$$

Examine the function 'f' for continuity and derivability at $x = 0$ where

$$f(x) = \begin{cases} 1 - x^2, & x \leq 0 \\ 1 + x^2, & x > 0 \end{cases}$$

$$L.H.L = R.H.L = F(a)$$

Continuous.

Continuity:-

L.H.L $f(x) = 1 - x^2$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 1 - x^2 = 1 - 0 = 1$$

$$f(x) = 1 - x^2$$

$$f(0) = 1 - 0 = 1$$

$$L.H.L = R.H.L = F(2)$$

$f(x)$ is Continuous.

R.H.L $f(x) = 1 + x^2$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 1 + x^2 = 1 + 0 = 1$$

L.H.D $F(x) = 1 - x^2$ at $x = 0$

$$f'(x) = \lim_{h \rightarrow 0} \frac{1 - (0+h)^2 - (1-0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{1} - h^2 - \cancel{1}}{h} = \lim_{h \rightarrow 0} -\frac{h^2}{h} = -h = 0$$

$F(0+h) = 1 - h^2$

$F(0) = 1 - 0 = 1$

$$f'(x) = \lim_{h \rightarrow 0} \frac{F(c+h) - F(c)}{h}$$

at $x = c$

R.H.D $F(x) = 1 + x^2$

$$f'(x) = \lim_{h \rightarrow 0} \frac{1 + (0+h)^2 - (1+0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{1} + h^2 - \cancel{1}}{h} = h = 0$$

$$f'_-(0) = f'_+(0)$$

$F(x)$ is diff.

Show that the function f defined as follows, is continuous at $x=2$, but not differentiable thereat :-

$$f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 - x, & 1 < x \leq 2 \\ 5x - 4, & x > 2 \end{cases}$$

$$f(x) = 2x^2 - x$$

$$f(2) = 2 \times 4 - 2 = 6$$

Continuous.

L.H.L $f(x) = 2x^2 - x$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} 2x^2 - x = 2(2)^2 - 2 = 2 \times 4 - 2 = 6$$

R.H.L

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 5x - 4 = \lim_{x \rightarrow 2^+} 5(2) - 4 = 10 - 4 = 6$$

$$\boxed{L.H.L = R.H.L = f(2)}$$

$f(x)$ is Continuous

$$\text{L.H.D } f(x) = 2x^2 - x$$



$$L f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(2+h)^2 - (2+h) - [2 \times 4 - 2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(4 + h^2 + 4h) - \cancel{2} - h - 8 + \cancel{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{8} + 2h^2 + 8h - h - \cancel{8}}{h} = \lim_{h \rightarrow 0} \frac{\cancel{h} [2h + 7]}{\cancel{h}} = 7$$

R.H.D $F(x) = 5x - 4$

$$RF'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5(2+h) - 4 - (5 \times 2 - 4)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{10} + 5h - \cancel{4} - \cancel{10} + \cancel{4}}{h} = 5$$

$$LF'(2) \neq RF'(2)$$

If the function f defined by $f(x) = \begin{cases} x^2 + 3x + a, & x \leq 1 \\ bx + 2, & x > 1 \end{cases}$ is derivable, then find the value of a and b .

$$L f'(x) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h)^2 + 3(1+h) + a - [1 + 3 + a]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{1} + h^2 + 2h + 3h + \cancel{3} + a - \cancel{4} - a}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h}[h+5]}{\cancel{h}} = 5$$

$$\begin{aligned}
 R F'(x) &= \lim_{h \rightarrow 0} \frac{F(1+h) - F(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{b(1+h) + 2 - (b+2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{b} + b\cancel{h} + \cancel{2} - \cancel{b} - \cancel{2}}{\cancel{h}} \\
 &= b
 \end{aligned}$$

$$L'F(x) = R'F(x)$$

$$5 = b$$

$$5 = 2 + a$$

$$a = 3$$

$$\begin{aligned}
 x^2 + 3x + a &= F(1) \\
 1 + 3 + a &= F(1) \\
 4 + a &= F(1)
 \end{aligned}$$

$$L.H.L = R.H.L = F(1)$$

$$R.H.L = b + 2$$

$$L.H.L = 1 + 3 + a = 4 + a$$

$$b + 2 = 4 + a$$

$$b = 2 + a$$

Exercise 5.5

i) If $f(x) = \sqrt{4 - x^2}$, $x \in (-2, 2)$ find $f'(x)$.

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$f'(x) = \frac{d}{dx} \sqrt{4 - x^2}$$

$$= \frac{d}{dx} (4 - x^2)^{1/2}$$

$$= \frac{1}{2} (4 - x^2)^{\frac{1}{2} - 1} \frac{d}{dx} (4 - x^2)$$

$$= \frac{1}{2\sqrt{4 - x^2}} (0 - 2x) = \frac{-x}{\sqrt{4 - x^2}} \text{ Ans}$$